Trial Examination 2006

VCE Mathematical Methods Units 3 & 4

Written Examination 1

Suggested Solutions

Question 1

a. $3a + 4a + 2a + a = 1$

$10a = 1$

$a = \frac{1}{10}$ A1

b. $\Pr(X \le 2) = (3 + 4 + 2) \times \frac{1}{10}$

$= \frac{9}{10}$ A1

c. $E(2X + 1) = E(2X) + E(1)$

$= 2E(X) + 1$ M1

$= 2 \times \Sigma x\Pr(X = x) + 1$

$= 2\left[0 \times \frac{3}{10} + 1 \times \frac{4}{10} + 2 \times \frac{2}{10} + 3 \times \frac{1}{10}\right] + 1$

$= 2(0.4 + 0.4 + 0.3) + 1$

$= 2 \times 1.1 + 1$

$= 3.2$ A1

Alternatively:

$E(2X + 1) = \Sigma(2x + 1)\Pr(X = x)$

$= (2 \times 0 + 1) \times 0.3 + (2 \times 1 + 1) \times 0.4 + (2 \times 2 + 1) \times 0.2 + (2 \times 3 + 1) \times 0.1$

$= 0.3 + 1.2 + 1.0 + 0.7$

$= 3.2$

Question 2

a. $f: y = e^{x+1} - 2 \quad \text{dom}_f: R, \text{ran}_f: (-2, \infty)$

$f^{-1}: x = e^{y+1} - 2$ M1

$e^{y+1} = x + 2$

$y + 1 = \log_e(x + 2)$

$y = \log_e(x + 2) - 1$

$f^{-1}(x) = \log_e(x + 2) - 1$ A1

b. $\text{dom } f^{-1} \Leftrightarrow \text{ran } f = (-2, \infty)$ A1

Question 3

a. $g(x) = 2x^2 + 4x - 7$

$= 2\left(x^2 + 2x - \frac{7}{2}\right)$

$= 2\left[(x^2 + 2x + 1) - 1 - \frac{7}{2}\right]$ M1

$= 2\left[(x+1)^2 - \frac{9}{2}\right]$

$= 2(x+1)^2 - 9$ A1

b. Dilation by a factor of two parallel to the y-axis (or from the x-axis). A1

Translation of one unit in the negative x direction (left) and nine units in the negative y direction (down). A1

Question 4

$\cos(3\pi x) = -\sin(3\pi x) \quad 0 \le x \le 1 \Rightarrow 0 \le 3\pi x \le 3\pi$

$\tan(3\pi x) = -1$ M1

The reference angle = $\frac{\pi}{4}$, with solutions in quadrants two and four.

$3\pi x = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 3\pi - \frac{\pi}{4}$ A1

$= \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$

$x = \frac{1}{4}, \frac{7}{12}, \frac{11}{12}$ A1

Question 5

a. $f(x) = x^2 e^{-3x}$

$f'(x) = 2xe^{-3x} + x^2(-3e^{-3x})$ (apply product rule) M1

$= xe^{-3x}(2 - 3x)$ A1

b. $f'(x) = 0$

$xe^{-3x}(2 - 3x) = 0$

$x = 0, \frac{2}{3}$

$f(0) = 0$ and $f\left(\frac{2}{3}\right) = \frac{4}{9}e^{-2} = \frac{4}{9e^2}$

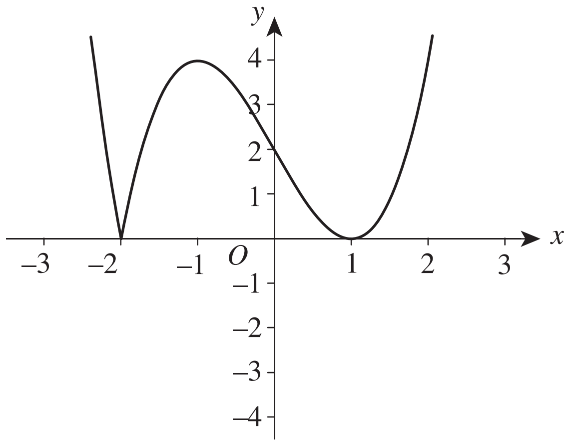
$\therefore$ stationary points at (0, 0) and $\left(\frac{2}{3}, \frac{4}{9e^2}\right)$

$4 \times \mathrm{A}\frac{1}{2}$ *for each x and y value*

rounded down to the nearest integer.

Question 6

a. $g(x) = |f(x)|$



A1

b. $R\backslash\{-2\}$ A1

c. $\text{Area} = \int_{-2}^{1} g(x)dx \quad \left(\text{or} = \left|\int_{-2}^{1} f(x)dx\right| \text{ etc.}\right)$

$$\begin{aligned} g(x) &= |f(x)| \\ &= -f(x) \text{ when } -2 \le x \le 1 \\ &= (x+2)(x-1)^2 \\ &= (x+2)(x^2-2x+1) \\ &= x^3 - 2x^2 + x + 2x^2 - 4x + 2 \\ &= x^3 - 3x + 2 \end{aligned}$$

A1

$$\therefore \text{ Area} = \int_{-2}^{1} (x^3 - 3x + 2)dx$$

$$= \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_{-2}^{1}$$

M1

$$= \left(\frac{1^4}{4} - \frac{3(1)^2}{2} + 2(1)\right) - \left(\frac{(-2)^4}{4} - \frac{3(-2)^2}{2} + 2(-2)\right)$$

$$= \frac{1}{4} - \frac{3}{2} + 2 - 4 + 6 + 4$$

$$= \frac{27}{4} \text{ square units}$$

A1

Question 7

a. Require $\int_0^2 k(x+1)\,dx = 1$ M1

$$k\int_0^2 (x+1)\,dx = 1$$

$$k\left[\frac{x^2}{2}+x\right]_0^2 = 1$$

$$k\left[\left(\frac{4}{2}+2\right)-(0+0)\right] = 1$$

$$4k = 1$$

$$k = \frac{1}{4}$$ A1

b. Require m such that $\int_0^m \frac{1}{4}(x+1)\,dx = \frac{1}{2}$ M1

$$\int_0^m (x+1)\,dx = 2$$

$$\left[\frac{x^2}{2}+x\right]_0^m = 2$$

$$\left(\frac{m^2}{2}+m\right)-(0+0) = 2$$

$$\frac{m^2}{2}+m = 2$$

$$m^2+2m = 4$$

$$m^2+2m-4 = 0$$ A1

$$m = \frac{-2\pm\sqrt{2^2-4(1)(-4)}}{2(1)}$$

$$= \frac{-2\pm\sqrt{20}}{2}$$

$$= \frac{-2\pm 2\sqrt{5}}{2}$$

$$= -1\pm\sqrt{5}$$

but $0 < m < 2 \quad \therefore\ m = -1+\sqrt{5}$ A1

Question 8

a. For $g(f(x))$ to exist, $\text{ran}_f \subseteq \text{dom}_g$.

dom_g is R.

$\therefore$ $\text{ran}_f \subseteq \text{dom}_g$ always $\therefore$ $g(f(x))$ exists. A1

b. $g(f(x)) = 1 - (3\sin(2x))^2$

$= 1 - 9\sin^2(2x)$ A1

Domain = dom_f (in this case): $[0, \pi]$ A1

c. $[-8, 1]$ A1

Question 9

$f(x) = x^{-\frac{1}{2}}$

$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$

$= -\frac{1}{2x\sqrt{x}}$ M1

$x = 4$, $h = -\frac{1}{10}$ A1

$f(3.9) = f\left(4 - \frac{1}{10}\right)$

$\approx f(4) - \frac{1}{10}f'(4)$

$= \frac{1}{\sqrt{4}} - \frac{1}{10}\left(-\frac{1}{2(4)\sqrt{4}}\right)$ M1

$= \frac{1}{2} + \frac{1}{(10)(8)(2)}$

$= \frac{1}{2} + \frac{1}{160}$

$= \frac{80}{160} + \frac{1}{160}$

$= \frac{81}{160}$ A1

Question 10

$$y = 4x^3 + 1$$

$$\frac{dy}{dx} = 12x^2$$

At $x = a$, the gradient of the tangent, $m_T = 12a^2$ M1

$\therefore$ the tangent has an equation of the form $y = 12a^2x + c$

The tangent passes through the origin $\therefore\ c = 0$

$$\therefore\ y = 12a^2x$$ A1

At $x = a$, $y = 4a^3 + 1$

$$\therefore\ 4a^3 + 1 = 12a^3$$ M1

$$8a^3 = 1$$

$$a^3 = \frac{1}{8}$$

$$a = \frac{1}{2}$$

$\therefore$ the tangent has the equation $y = 12\left(\frac{1}{2}\right)^2 x$

$$y = 3x$$ A1